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ON THE BANACH SPACE ISOMORPHISM TYPE OF AF C*-ALGEBRAS AND THEIR TRIANGULAR SUBALGEBRAS

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ABSTRACT

It is shown that all the approximately finite dimensional C*-algebras which are not of Type I are isomorphic as Banach spaces. This generalizes the matroid case given previously by Arazy. Analogous results are obtained for various families of triangular subalgebras of AF C*-algebras. In addition the classification of various continua of Type I AF C*-algebras is discussed.

A C*-algebra is approximately finite (or AF) if it is a closed union of finite-dimensional C*-subalgebras. Those AF C*-algebras for which the finite-dimensional subalgebras can be taken to be full matrix algebras are known as matroid C*-algebras. J. Arazy [1] has shown that with the exception of the algebra $\mathcal{K}$ of compact operators, which is the unique matroid algebra with separable dual, all (infinite-dimensional) matroid C*-algebras are isomorphic as Banach spaces. We generalise this by showing that all the AF C*-algebras which are not of Type I are isomorphic. In particular we see that a simple AF C*-algebra is either isomorphic to $\mathcal{K}$ or to the Fermion algebra F, the unital matroid C*-algebra $\varinjlim M_{2^k}$. We also comment on the classification of AF C*-algebras of Type I.

A similar analysis is given for various distinguished triangular subalgebras of AF C*-algebras and it is this non-self-adjoint context that motivated the present study.

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In particular the refinement limit algebras $\varinjlim (T_{n_k}, \rho_k)$ are all isomorphic to the model algebra $\mathcal{T}_{2^\infty} = \varinjlim (T_{2^k}, \rho_k)$, and the standard limit algebras are all isomorphic to the model algebra $\mathcal{S}_{2^\infty} = \varinjlim (T_{2^k}, \sigma_k)$. From this we are able to deduce that all the (proper) alternation algebras are isomorphic. It seems probable that $\mathcal{T}_{2^\infty}$ and $\mathcal{S}_{2^\infty}$ are not isomorphic. Some evidence in support of this (see Remark 2.4) is that there are no "natural" complemented contractive injections $F \to \mathcal{S}_{2^\infty}$ and for this reason the methods of this paper cannot be applied. In fact it seems likely that non-self-adjoint subalgebras of AF C*-algebras provide an interesting diversity of Banach spaces. This is to be expected in view of the analogies that exist between triangular algebras and function spaces. Also, in an associated nonselfadjoint context, Arias [3] has recently shown that there are nonisomorphic nest subalgebras (perhaps uncountably many) in the trace class.

The proofs below are straightforward and self-contained. In particular we do not require the C*-algebraic classification of the matroid algebras or the AF C*-algebras (given in Dixmier [7] and Elliott [8] respectively). The key step in the self-adjoint case is to show that if B is an AF C*-algebra whose Bratteli diagram has a particular property, which we call the Fermion property, then for any other AF C*-algebra A there exists a complemented linear contractive injection $\gamma: A \to B$. This injection is obtained by constructing an infinite commuting diagram whilst at the same time constructing a commuting system of left inverses for the partial injections. This ensures that in the limit the closed space $\gamma(A)$ is complemented in terms of a completely contractive linear map.

This form of explicit construction lends itself to natural modifications to deal with triangular subalgebras of AF C*-algebras.

As this paper was being prepared Simon Wasserman pointed out to the author that the isomorphism of non Type I AF algebras has also been obtained in a different way by Kirchberg as a consequence of his work on nuclearly embeddable C*-algebras and exact C*-algebras. In fact, Kirchberg deduces that all separable nuclear non Type I C*-algebras are isomorphic. The rather deep result that leads to this isomorphism is that separable unital nuclear C*-algebras are unitally completely isometrically embeddable in the Fermion algebra. The proof of this, as well as the simplified proof given by Wasserman [18], is quite involved with C*-algebra theory and contrasts markedly with our approach for the simpler case of AF C*-algebras.

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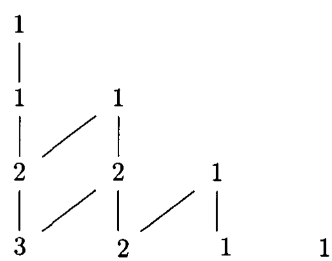
1. AF algebras with The Fermion property

The Fermion algebra (or CAR algebra) is the unital matroid C*-algebra $F = \varinjlim M_{2^k}$ with Bratteli diagram

A general AF C*-algebra, with presentation $A = \varinjlim A_k$, has an associated Bratteli diagram in which, similarly, multiple edges between two vertices indicate, through their multiplicity, the multiplicity of the partial embedding between the summands associated with the vertices. We shall say that the Bratteli diagram has the **Fermion property** if there is a sequence of vertices $v_1, v_2, \ldots$ associated with summands of $A_{n_1}, A_{n_2}, \ldots$, respectively, with n_k an increasing sequence, such that there is more than one vertical path between each consecutive pair v_i, v_{i+1}. That is, there is a sequence of *partial* embeddings $A_{n_k} \to A_{n_{k+1}}$, with nonzero compositions, each of multiplicity at least two. Thus, the Pascal triangle Bratteli diagram has the Fermion property, whereas the following diagram does not:



We now show that two AF C*-algebras whose Bratteli diagrams have the Fermion property are linearly homeomorphic. The next lemma, which is analogous to Lemma 2.11 of [1], is the key result required in the proof.

Let A and B be finite-dimensional C*-algebras with chosen matrix unit systems $\{e_{ij}\colon (i,j) \in I\}$ and $\{f_{ij}\colon (i,j) \in J\}$ respectively. Assume that I, J are block diagonal subsets of $\{1,\ldots,m\}^2$ for some m. Let $a = (a_{ij})$ belong to A. A linear map $\gamma\colon A \to B$ is said to be of **compression type** with respect to these systems if γ is a (block diagonal) direct sum of maps of the form $\alpha \circ \beta$ where $\beta\colon A \to M_n$ is given by

$$\beta((a_{ij})) = (a_{k_s,k_t})_{s,t=1}^n,$$

where $\{k_1,\ldots,k_n\}^2 \subseteq I$, and where $\alpha\colon M_n \to B$ is a multiplicity one algebra injection of the form

$$\alpha((b_{s,t})) = (b_{l_s,l_t})_{s,t=1}^n$$

where $\{l_1,\ldots,l_n\}^2 \subseteq J$. If, additionally, $\{l_1,\ldots,l_n\}$ and $\{t_1,\ldots,t_n\}$ are ordered subsets ($l_1 < l_2$ etc), then we refer to γ as an **ordered compression type map**.

Let $A = A_1 \oplus \cdots \oplus A_r$ where $A_1,\ldots,A_r$ are the matrix algebra summands of A and let $\gamma\colon A \to B$ be a map of compression type, as above, with $\gamma = \gamma_1 \oplus \cdots \oplus \gamma_p$, where $\gamma_1,\ldots.\gamma_p$ are the elementary summands of γ. The map γ has isometric restriction to A_1 if (and only if) there is a summand γ_i which is isometric on A_1. It follows that γ is isometric if (and only if) the summands can be reordered and relabelled as $(\gamma_1 \oplus \cdots \oplus \gamma_r) \oplus (\gamma_{r+1} \oplus \cdots \oplus \gamma_p)$ so that $\gamma' = \gamma_1 \oplus \cdots \oplus \gamma_r$ is a (multiplicity one) algebra injection of A into B. With such a relabelling there is an associated contractive left inverse map $\delta\colon B \to A$, of compression type, satisfying $\delta \circ \gamma = id_A$. The map δ is the composition of compression onto the range of γ', followed by the inverse map of γ' restricted to its range.

Changing notation, let $A = \varinjlim (A_k, \phi_k)$, and $B = \varinjlim (B_k, \psi_k)$ be presentations of the AF C*-algebras A and B where the maps ϕ_k and ψ_k are isometric C*-algebra injections. Assume that the matrix unit systems $\{e_{ij}^k\colon (i,j) \in I_k\}$ and $\{f_{ij}^k\colon (i,j) \in J_k\}$ have been chosen for A_k and B_k respectively, so that each map ϕ_k and ψ_k maps matrix units to sums of matrix units, and, for the moment, assume that A is unital and that the embeddings ϕ_k are unital. Assume furthermore that the Bratteli diagram for the system $\{B_k, \psi_k\}$ has the Fermion property. By composing maps, forming a subsystem from such compositions, and relabelling, we may assume that $B_k = M_{r_k} \oplus B_k'$ and that the partial embedding of ψ_k from M_{r_k} into $M_{r_{k+1}}$ has multiplicity at least two.

LEMMA 1.1: *With the assumptions above there is a commuting diagram*

$$\begin{array}{ccccc} A_1 & \xrightarrow{\phi_1} & A_2 & \xrightarrow{\phi_2} & \\ {\scriptstyle\gamma_1}\downarrow & & {\scriptstyle\gamma_2}\downarrow & & \cdots \\ B_{n_1} & \xrightarrow{\theta_1} & B_{n_2} & \xrightarrow{\theta_2} & \end{array}$$

where each map γ_k is an isometric linear map of compression type relative to the given matrix unit system, and where $\theta_1, \theta_2, \ldots$ are compositions of the given embeddings $\psi_1, \psi_2, \ldots$. Furthermore there are linear contractions $\delta_k \colon B_{n_k} \to A_k$, of compression type, satisfying $\delta_k \circ \gamma_k = id$, such that the diagram

$$\begin{array}{ccccc} A_1 & \xrightarrow{\phi_1} & A_2 & \xrightarrow{\phi_2} & \\ {\scriptstyle\delta_1}\uparrow & & {\scriptstyle\delta_2}\uparrow & & \cdots \\ B_{n_1} & \xrightarrow{\theta_1} & B_{n_2} & \xrightarrow{\theta_2} & \end{array}$$

commutes. In particular there exists an isometric injection $\gamma \colon A \to B$ and a contractive map $\delta \colon B \to A$ such that $\gamma \circ \delta$ is a contractive projection onto the range of γ.

Proof: Using the Fermion property for B choose n_1 large enough so that there exists a multiplicity one linear isometry $\gamma_1 \colon A_1 \to B_{n_1}$, of compression type, with range in the summand M_{r_1} of the decomposition $B_{n_1} = M_{r_1} \oplus B'_{n_1}$. We may assume that γ_1 has the form $\gamma_1(a) = [a \oplus 0_*] \oplus \{0\}$, that is that the partial embedding of γ_1 from A_1 into M_{r_1} has a proper zero summand 0_*. (We indicate the distinguished first summand of B_{n_2} with square brackets and the remaining summands are grouped in braces.) For $n_2 > n_1$, to be chosen, the composed map $\theta_1 \colon B_{n_1} \to B_{n_2}$ has partial embeddings $\sigma_1 \colon M_{r_1} \to M_{r_2}$, $\tau_1 \colon B'_{n_1} \to M_{r_2}$, $\tau_2 \colon B_{n_1} \to B'_{n_2}$, and by relabelling the matrix units of M_{r_2} we may assume that σ_1 is of standard type, that is,

$$\sigma_1(b) = b \oplus \cdots \oplus b \oplus 0 \qquad (b \text{ appearing } t \text{ times}).$$

(The zero summand may be absent.) Using the Fermion property hypothesis we may choose n_2 so that t is arbitrarily large. Note that the composition $\theta_1 \circ \gamma_1$ has the form

$$a \to [\sigma_1(a \oplus 0_*) \oplus \tau_1(0)] \oplus \{\tau_2(\gamma_1(a))\}.$$

Consider now the given map $\phi_1\colon A_1 \to A_2$. Let

$$A_1 = A_{1,1} \oplus \cdots \oplus A_{1,p}, \quad A_2 = A_{2,1} \oplus \cdots \oplus A_{2,q}$$

be the matrix algebra decompositions. Relabelling matrix units of A_2 we may assume that ϕ_1 is given in a standard form with respect to the matrix unit systems, that is,

$$\phi_1\colon A_{1,1} \oplus \cdots \oplus A_{1,p} \to A_{2,1} \oplus \cdots \oplus A_{2,q}$$

where the summand of $\phi_1(a_1 \oplus \cdots \oplus a_p)$ in the matrix summand $A_{2,t}$ is

$$(\sum_1^{k_{1,t}} \oplus a_1) \oplus \cdots \oplus (\sum_1^{k_{p,t}} \oplus a_p),$$

with the understanding that some of these summands may be absent. The integer $k_{s,t}$, for $1 \le s \le p, 1 \le t \le q$, is the multiplicity of the partial embedding for ϕ_1 from $A_{1,s}$ to $A_{2,t}$.

We now construct $\gamma_2\colon A_2 \to B_{n_2}$ as an isometric linear multiplicity one injection of compression type, as suggested by the following diagram:

$$\begin{array}{ccc} a & \xrightarrow{\phi_1} & [\sum_{1,1} \oplus \cdots \oplus \sum p, 1] \oplus \cdots \oplus [\sum_{1,q} \oplus \cdots \oplus \sum_{p,q}] \\ \gamma_1 \downarrow & & \gamma_2 \downarrow \\ [a \oplus 0_*] \oplus 0 & \xrightarrow{\theta_1} & [\sigma_2(a \oplus 0_*) \oplus \tau_1(0)] \oplus \tau_2(\gamma_1(a)) \end{array}$$

Using the Fermion property choose n_2 large enough so that

$$t \ge k_{i,1} + \cdots + k_{i,q} \quad , 1 \le i \le p.$$

These inequalities guarantee that there exists a one to one correspondence of the summands of $\phi_1(a)$ with *some* of the appropriate nonzero summands of $\sigma_1(a \oplus 0_*)$. The point of this observation is that there is a natural multiplicity one algebra injection γ_2' from A_2 to M_{r_2} (of compression type) which respects this correspondence. Construct γ_2 by adding extra summands to γ_2' to obtain a linear isometry, of compression type, satisfying $\gamma_2(\phi_1(a)) = \theta_1(\gamma_1(a))$ for all a in A_1. Since ϕ_1 is isometric this is possible.

Define $\delta_1 = \gamma_1^{-1} \circ \eta_1$ where η_1 is the compression map onto the range of γ_1, noting that γ_1^{-1} is well-defined on this range. Similarly define δ_2 in terms of γ_2. Thus, $\delta_2 = (\gamma_2)^{-1} \circ \eta_2$ where η_2 is the compression onto the range of γ_2. To see the important equality $\phi_1 \circ \delta_1 = \delta_2 \circ \theta_1$, let $b \in B_{n_1}$, and let $\eta_1(b) = [a \oplus 0_*] \oplus 0$. By the construction of γ_2 and δ_2 we have

$$\delta_2(\theta_1(b)) = \delta_2(\theta_1(\eta_1(b))).$$

This is because the domain of δ_2 is *subordinate* to the nonzero summands of $\sigma_1(a \oplus 0_*)$, and $\theta_1(b - \eta_1(b))$ vanishes on these summands. Thus

$$\delta_2(\theta_1(b)) = \delta_2(\theta_1(\eta_1(b))) = \delta_2(\theta_1([a \oplus 0_*] \oplus 0)) = \phi_1(a) = \phi_1(\delta_1(b)).$$

Repeating the arguments above obtain inductively isometric maps $\gamma_3, \gamma_4, \ldots$, with distinguished contractive left inverses $\delta_3, \delta_4, \ldots$. Indeed, note that after relabelling the matrix units of B_{n_2} the map γ_2: $A_2 \to M_{r_2} \oplus B'_{n_2}$ can be written in the form

$$a \to [a \oplus 0 \oplus \gamma_{2,1}(a)] \oplus (\gamma_{2,2}(a)),$$

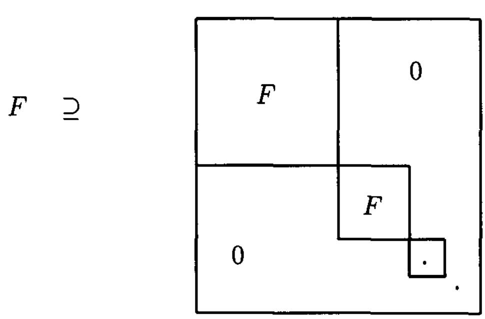
and so the construction of $\gamma'_3, \delta_3, \gamma_3$ is obtained in exactly the same way as $\gamma'_2, \delta_2, \gamma_2$. ∎

The lemma shows that there is a complemented isometric linear injection $A \to B$. If A is not unital then A has a complemented isometric linear injection into its unitisation $A^\sim$, and so the nonunital case follows on consideration of the composition $A \to A^\sim \to B$.

The next lemma also appears in Arazy's paper. For completeness we give a proof. Write $X \approx Y$ if X and Y are linearly homeomorphic Banach spaces.

LEMMA 1.2: $F \approx c_0(F)$.

Proof: Realise F as the direct limit $\varinjlim (M_{2^k}, \rho_k)$ where ρ_k: $a \to a \otimes I_2$. Define a natural injection β: $c_0(F) \to F$ which is suggested by the following inclusion diagram.



More precisely let F_0 be the subspace $\varinjlim (M^0_{2^k}\rho_k)$, of codimension one given by the subsystem determined by the subspaces

$$M^0_{2^k} = \{(a_{ij}) \in M_{2^k}\colon a_{2^k,2^k} = 0\}.$$

Define $c_0(F) \to F_0$ as follows. Identify the first copy of F in $c_0(F) = F \oplus F \oplus \cdots$ with $p_1 F_0 p_1$ where $p_1 = e_{1,1}$ in M_2. (Identify $e_{1,1}$ with its image in the limit.) Identify the second copy with $p_2 F_0 p_2$, where $p_2 = e_{33}$ in M_{2^2}, and so on. The resulting inclusion $c_0(F) \to F$ has range which is the range of the projection $E\colon F_0 \to F_0$ given by $E(a) = \lim_k(\sum_{j=1}^k p_j a p_j)$. Thus $c_0(F)$ is complemented in F_0, and hence in F. Thus $F \approx c_0(F) \oplus X \approx c_0(F) \oplus c_0(F) \oplus X \approx c_0(F) \oplus F \approx c_0(F)$. ∎

The proof of the next theorem now reduces to a routine application of the Pelczynski decomposition method.

THEOREM 1.3: *Let A and B be AF C*-algebras given by direct systems whose Bratteli diagrams have the Fermion property. Then A and B are isomorphic as topological vector spaces.*

Proof: We may assume that $B = F$. By Lemma 1.1, and the remarks concerning the unital case, there exist contractive injective complemented maps $A \to F$ and $F \to A$. Thus, by Lemma 1.2,

$$c_0(A) \to c_0(F) \approx F \to A.$$

Hence, just as with F, we have $A \approx c_0(A) \oplus Y \approx c_0(A) \oplus c_0(A) \oplus Y \approx c_0(A) \oplus A \approx c_0(A)$.

Consider now the fact that $A \approx F \oplus Z$ for some closed subspace Z of A, and obtain $A \approx F \oplus Z \approx c_0(F) \oplus Z \approx c_0(F) \oplus F \oplus Z \approx F \oplus A$. Similarly, $F \approx F \oplus A$, and so $F \approx A$. ∎

Let $A = \varinjlim A_k$ be an (infinite-dimensional) AF C*-algebra, with a corresponding Bratteli diagram, which is simple. This means that for each vertex v of the diagram, at level k, there is a lower level m such that there exist downward paths from v to *all* the vertices at level m. (See Bratteli [5].) In particular any two vertices at a given level have downward paths that meet in a common vertex. This weaker property is precisely the Bratteli diagram criterion for the triviality of the centre of A. Suppose additionally, that the Bratteli diagram fails to have the Fermion property. Then there must exist a vertex with a *unique* downward path. For otherwise there is repeated branching and convergence characteristic of the Fermion property.

The unique downward path determines a subsystem of A which defines a subalgebra J which is isomorphic to $\mathcal{K}$ or M_n for some n. Since J is in fact an ideal, and A is simple, it follows that $A = \mathcal{K}$. Thus we have obtained

COROLLARY 1.4: *Let A be a simple (infinite-dimensional) approximately finite C^*-algebra. Then $A \approx \mathcal{K}$ or $A \approx F$.*

A C*-algebra is said to be of Type I if its star representations generate Type I von Neumann algebras. Also it is known that this is equivalent to the apparently weaker assertion that factorial star representations are Type I. (See, for example, [9].) Using this we can strengthen the last corollary.

COROLLARY 1.5: *Let A be an approximately finite C^*-alegbra which is not Type I. Then A is isomorphic to F as a linear topological vector space.*

Proof: Let $A = \varinjlim A_k$, with Bratteli diagram without the Fermion property. We show that the factorial representations of A are Type I. Note first that if $\pi\colon A \to L(H)$ is a factorial representation, then $\ker \pi$ is an ideal, and $A/\ker \pi$ is an AF C*-algebra with Bratteli diagram obtained as a subdiagram of the diagram for A. (See Bratteli [5].) Since this subdiagram also fails to have the Fermion property we may as well assume that $\ker \pi = \{0\}$. With this assumption it follows that the centre of A must be trivial. By the argument preceding Corollary 4 it follows that A possesses an ideal which is isomorphic to $\mathcal{K}$ and which is associated with a vertex of the diagram that has a unique descending path. Let p be a minimal projection in the matrix summand corresponding to this vertex. Then p is minimal in A, and so $\pi(p)$ is minimal in $\pi(A)''$. Thus the factor $\pi(A)''$ is Type I. ∎

TYPE I AF C*-ALGEBRAS. We comment on the isomorphism types of the (infinite dimensional) Type I AF C*-algebras.

In the separable dual case the following three algebras present themselves:

(i) c_0, the space of diagonal compact operators,

(ii) $\mathcal{R} = (\sum_{k=1}^{\infty} \oplus M_k)_{c_0}$,

(iii) $\mathcal{K}$, the compact operators, and they are known to be pairwise nonisomorphic. See Arazy and Lindenstrauss [2]. We can distinguish (i) from (ii) and (iii) by noting that all bounded maps from c_0 to ℓ^2 are 2-summing. A simple proof can be found in Pisier's notes [13]. On the other hand matrix realisations provide "top row" maps $\mathcal{K} \to \ell^2$, $\mathcal{R} \to \ell^2$ which are not 2-summing. Alternatively, it can be seen in Hamana [11] and Chu and Iochum [6] that (i) has the Dunford Pettis property whereas (ii) and (iii) do not. To distinguish $\mathcal{R}$ and $\mathcal{K}$ one can note that the first space has a dual space with the Schur property, (that weakly convergent sequences are norm convergent) whereas the trace class operators do not. See [2], [11] and [6] for further details.

If an AF C*-algebra has a separable dual space then it is easy to see that it does not have the Fermion property and this limits the possibilities for the type of Bratteli diagram. The Bratteli diagrams in this case fall naturally into three types, namely, Type (i), in which there is a uniform bound on the sizes of the matrix summands, Type (ii), in which for every path in the diagram there is a uniform bound on the sizes of the associated summands, and Type (iii), being the rest. In the particular context of a diagram with finite width the associated algebra has a finite composition series and from this it can be shown to be isomorphic to $\mathcal{K}$ as a Banach space.

Bessaga and Pelczynski [4] have classified the spaces $C(S)$ with S countable. For each isomorphism type there is a countable ordinal α with $C(S) \approx C(\alpha)$, and $C(\beta) \approx C(\alpha)$ if and only if $\alpha \leq \beta < \alpha^\omega$ or $\beta \leq \alpha < \beta^\omega$. In particular there is a continuum of isomorphism types of abelian AF C*-algebras with diagrams of type (i). One would expect there to be similar continua for the algebras whose diagrams are of type (ii) and type (iii).

Turning to the algebras whose dual space is not separable there are, in the first instance, six natural C*-algebras to consider.

(iv) $C(K)$, with K a Cantor space,

(v) $C(K) \oplus \mathcal{R}$,

(vi) $C(K) \otimes \mathcal{R}$,

(vii) $C(K) \oplus \mathcal{K}$,

(viii) $(C(K) \otimes \mathcal{R}) \oplus \mathcal{K}$,

(ix) $C(K) \otimes \mathcal{K}$.

These algebras are associated with six different types of Bratteli diagram. Thus, all diagram types have uncountably many paths. For Type (iv) there is a uniform bound on the sizes of all matrix summands. For Type (v) there is no uniform bound, but for each n there are at most countably many paths on which the matrix sizes exceed n, and there is no path with unbounded matrix sizes. A Type (vi) diagram is not of Type (iv) or (v) and there are no paths with unbounded matrix sizes. For Type (vii) there do exist paths of unbounded matrix size and for some n there are at most countably many paths whose matrix sizes exceed n. A Type (viii) diagram has only countably many "unbounded paths", but for each n there are uncountably many paths whose matrix sizes exceed n. A Type (ix) diagram has uncountably many unbounded paths.

It seems plausible that isomorphic AF C*-algebras have Bratteli diagrams of the same diagram type, and within some of these diagram types ((v), (vii), (viii)) one would expect there to be added ordinal type complexities as in the separable dual case. For example one might expect there to be a continua of Banach space types of the form $C(K) \oplus (C(\alpha) \otimes \mathcal{K})$.

With regard to the duals of AF C*-algebras, Wojtaszczyk has shown that there are just three separable duals, namely the duals of the algebras (i), (ii), and (iii). It seems reasonably to conjecture that there are precisely nine dual spaces of Type I AF C*-algebras.

2. Triangular subalgebras

There are three well-known families of triangular subalgebras of UHF C*-algebras, namely the refinement algebras $\varinjlim (T_{n_k}, \rho_k)$, the standard algebras $\varinjlim (T_{n_k}, \sigma_k)$ and the alternation algebras, $\varinjlim (T_{n_k}, \alpha_k)$. The (unital) embeddings determining these limits have the form

$$\rho_k((a_{ij})) = (a_{ij} I_{t_k}), \quad \sigma_k(a) = I_{t_k} \otimes a,$$

where t_k is the multiplicity of the embedding, and in the alternation case α_k alternates between these two types. We shall prove the following theorem.

THEOREM 2.1:

(i) *The standard limit algebras are isomorphic as Banach spaces.*

(ii) *The refinement limit algebras are isomorphic as Banach spaces.*

(iii) *The alternation limit algebras are isomorphic as Banach spaces.*

Another well-known class consists of the various "refinement with twist" limits $\varinjlim(T_{n_k}, \tau_k)$, where τ_k agrees with ρ_k on all the standard matrix units e_{ij} of T_{n_k}, with the exception of those superdiagonal matrix units in the last column. For these

$$\tau_k(e_{i,n_k}) = e_{i,n_k} \otimes u_k$$

where u_k is a permutation unitary in M_{t_k}. It was shown in Hopenwasser and Power [12] that these algebras provide uncountably many algebra isomorphism classes, distinct from the refinement limits. On the other hand we have

THEOREM 2.2: *If A is a refinement with twist algebra, as above, then as a Banach space, A is isomorphic to the model refinement algebra $\mathcal{T}_{2^\infty}$.*

The algebras above are examples of (canonical regular) triangular subalgebras of UHF C*-algebras. In a different direction one can generalize the standard embedding limit algebras by considering ordered Bratteli diagrams ([15], [14]). A typical such embedding has the form

$$\beta\colon T_{q_1} \oplus \cdots \oplus T_{q_s} \to T_{p_1} \oplus \cdots \oplus T_{p_r}$$

where

$$\beta\colon a_1 \oplus \cdots \oplus a_s \to (\sum_{j=1}^{t_1} \oplus b_{1,j}) \oplus \cdots \oplus (\sum_{j=1}^{t_r} \oplus b_{r,j})$$

and where each $b_{k,l}$ is one of the summands $a_1, \ldots, a_s$. The summations here mean block diagonal direct sums. In the nonunital case one also allows the $b_{k,l}$ to be zero summands. For a simple example, consider the embedding β_1 from $T_2 \oplus T_3 \oplus T_4$ to $T_7 \oplus T_6 \oplus T_5$ given by

$$\beta_1\colon a \oplus b \oplus c \to (b \oplus c) \oplus (a \oplus c) \oplus (a \oplus b).$$

This embedding is not inner conjugate to

$$\beta_2\colon a \oplus b \oplus c \to (b \oplus c) \oplus (c \oplus a) \oplus (a \oplus b)$$

and the difference can be indicated by ordered Bratteli diagrams. A consequence of this diversity is the existence of uncountably many nonisomorphic limit algebras with the same generated C*-algebra. Once again, however, they correspond to a unique Banach space type.

THEOREM 2.3: *Let A be a triangular limit algebra determined by an ordered Bratteli diagram which has the Fermion property. Then, as a Banach space, A is isomorphic to the model algebra $\mathcal{S}_{2^\infty}$.*

Recall the definition of a linear map $\gamma\colon M_n \to M_m$ which is of *ordered compression type* and note that such a map has a restriction $\gamma\colon T_n \to T_m$. Using direct sums of such maps define *ordered compression type maps* $T_q \to T_{p_1} \oplus \cdots \oplus T_{p_r}$ and use these to define general *ordered compression type maps* $\gamma = T_{q_1} \oplus \cdots \oplus T_{q_s} \to T_{p_1} \oplus \cdots \oplus T_{p_r}$. If $\gamma(a_1 \oplus \cdots \oplus a_s)$ has at least one complete summand a_i, for each i, then γ is isometric. In this case it follows, as in section 2, that γ has an associated left inverse.

The Proof of Theorem 2.3: Note first that the conclusions of Lemma 1.1 hold in the triangular context wherein we make the following new assumptions: $A = \varinjlim(A_k, \varphi_k)$ and $B = \varinjlim(B_k, \psi_k)$ are limit algebras determined by ordered Bratteli diagrams and the diagram for B has the Fermion property. To adapt the proof we use ordered compression maps in place of compression maps. The argument is virtually the same but notationally awkward since we cannot make simplifying reorderings of summands by relabelling matrix units in the codomain. The first occasion for this is the expression for $\theta_1 \circ \gamma_1(a)$. (The map γ_1 can be chosen to be of ordered compression type.) However $\theta_1 \circ \gamma_1(a)$ is an ordered direct sum such that, in the first summand T_{r_2} of B_{n_2}, for suitably large n_2, there appear many summands which are copies of the summands of a. Thus, as before, there is an association of the summands of $\phi_1(a)$ with some of the summands of $\theta_1 \circ \gamma_1(a)$, if n_1 is large enough. However we assume additionally that this association respects the order in which summands of $\phi_1(a)$ appear in each summand $A_{2,i}$. As before define a multiplicity one injection γ_2' of ordered compression type, from A_2 to B_{n_2}, which respects this correspondence. The extension γ_2 of γ_2' and the left inverse δ_2 of γ_2 (and γ_2') are defined as before, and once again the desired commuting diagrams follow upon iterating this procedure. In the case that A is nonunital obtain a complemented unital isometric injection $A \to B$ by considering the unitisation of A.

The argument of Lemma 1.2 also serves to give a natural complemented injection $c_0(\mathcal{S}_{2^\infty}) \to \mathcal{S}_{2^\infty}$. (The diagram of Lemma 1.2, however, is not appropriate for standard embeddings.) The remainder of the proof follows as before. ∎

The proof of Theorem 2.1 (ii): This follows a similar scheme. Let $A = \varinjlim(A_k, \phi_k)$ be a refinement limit algebra and let $\mathcal{T}_{2^\infty} = \varinjlim(B_k, \psi_k)$ be the 2^∞ refinement limit algebra. To obtain the appropriate version of Lemma 1.1 first choose n_1 large enough so that there is a multiplicity one ordered compression type map γ_1: $A_1 \to B_{n_1}$ given by $\gamma_1(a) = a \oplus 0$ (block diagonal direct sum). The matrix $\phi_1(a)$ has the form $(a_{ij}I_t)$ where t is the multiplicity of ϕ_1. Choose θ_1: $B_{n_1} \to B_{n_2}$, a composition of consecutive maps $\psi_1, \psi_2, \ldots$ so that the multiplicity of θ_1 exceeds that of ϕ_1. Thus

$$\phi_1((a_{ij})) = (a_{ij}I_t)$$

and

$$\theta_1(\gamma_1((a_{ij}))) = (a_{ij}I_s) \oplus 0$$

where $s > t$. There is now natural multiplicity one isometric (algebra) injection γ_2': $A_2 \to B_{n_2}$ with the property that

$$\gamma_2'(a_{ij}I_t) = (a_{ij}(I_t \oplus 0_{s-t})) \oplus 0.$$

This (as before) does not yet give a commuting square. Nevertheless we can add multiplicity one summands of ordered compression type to create γ_2, an extension of γ_2', so that $\gamma(a_{ij}I_t) = (a_{ij}I_s) \oplus 0$, and thus we obtain the first commuting square of Lemma 1.1 in this context. Furthermore, γ_1 has a natural left inverse δ_1, and, as before, γ_2' can be used in the definition of a left inverse δ_2 for γ_2, which extends δ_1 in the obvious way. The construction of the desired maps $\gamma_3, \delta_3, \gamma_4, \delta_4, \ldots$ is obtained similarly.

We have $\mathcal{T}_{2^\infty} \approx c_0(\mathcal{T}_{2^\infty})$, by the argument of Lemma 1.2 (the diagram is appropriate this time), and the proof is completed as before. ∎

The proof of Theorem 2.1 (iii): Let s and r be the generalised integers associated with the triangular limit algebras $\mathcal{S}_s$ and $\mathcal{T}_r$. Consider the subalgebra $\mathcal{S}_s \star \mathcal{T}_r$ of $B_s \otimes B_r$ given by

$$\mathcal{S}_s \star \mathcal{T}_r = (\mathcal{S}_s \cap \mathcal{S}_s^*) \otimes \mathcal{T}_r \;+\; (\mathcal{S}_s^0 \otimes B_r)$$

where $B_s = C^*(\mathcal{S}_s)$ and $B_r = C^*(\mathcal{T}_r)$ are the generated C*-algebras, and where $\mathcal{S}_s^0$ is the strictly upper triangular subalgebra of $\mathcal{S}_s$. In the terminology of [16] and [17] this algebra is the lexicographic product of the ordered pair $\mathcal{S}_s, \mathcal{T}_r$. If s and r are not finite then this product coincides with the proper alternation algebra for the pair r, s. This formula forms the basis of the proof.

Note first that $\mathcal{S}_s \star \mathcal{T}_r$ is bicontinuously isomorphic to the ℓ^∞ direct sum of the component spaces above. Also $\mathcal{S}_s \cap \mathcal{S}_s^*$ is C*-algebraically isomorphic to $C(K)$, where K is a Cantor space. From the linear isomorphism $\mathcal{T}_r \approx \mathcal{T}_{2^\infty}$ we obtain

$$(\mathcal{S}_s \cap \mathcal{S}_s^*) \otimes \mathcal{T}_r \approx C(K) \otimes \mathcal{T}_r \approx C(K) \otimes \mathcal{T}_{2^\infty} \approx (\mathcal{S}_s \cap \mathcal{S}_s^*) \otimes \mathcal{T}_{2^\infty}.$$

It remains then to show that $\mathcal{S}_r^0 \otimes B_r$ and $\mathcal{S}_{2^\infty} \otimes B_{2^\infty}$ are bicontinuously isomorphic. However the maps of Lemma 1.1 and its non-self-adjoint variants respect tensor products. For example, the isometric map $\gamma: A \to B$ of Lemma 1.1 also provide complemented isometric injections $\gamma \otimes id: A \otimes D \to B \otimes D$, where D is a closed subspace of an AF C*-algebra (for example) and the tensor product is the injective, or spatial, tensor product. And so we obtain the needed equivalences

$$\mathcal{S}_r^0 \otimes B_r \approx \mathcal{S}_{2^\infty}^0 \otimes B_r \approx \mathcal{S}_{2^\infty}^0 \otimes B_{2^\infty}$$

and the proof follows. ∎

The proof of Theorem 2.2: Let $A = \varinjlim(T_{n_k}, \tau_k)$ be a refinement with twist limit algebra. Let $A_0 = \varinjlim(T_{n_k}^-, \tau_k) = \varinjlim(T_{n_k}^-, \rho_k)$ where $T_{n_k}^- = \operatorname{span}\{e_{i,j}: j < N_k\}$. Also, let $\mathcal{T} = \varinjlim(T_{n_k}, \rho_k)$ be the associated refinement algebra. Then $T_{n_k} \approx T_{n_k}^- \oplus \ell_{n_k}^2$. That is, the map

$$a \to a(1 - e_{n_k,n_k}) \oplus ae_{n_k,n_k}$$

is a bicontinuous linear isomorphism to the ℓ^∞ direct sum. From the commuting diagram

$$\begin{array}{ccccc} T_{n_1} & \xrightarrow{\tau_1} & T_{n_2} & \xrightarrow{\tau_2} & \cdots \\ \downarrow & & \downarrow & & \\ T_{n_1}^- \oplus l_{n_1}^2 & \longrightarrow & T_{n_2}^- \oplus l_{n_2}^2 & \longrightarrow & \end{array}$$

we obtain a bicontinuous isomorphism $A \approx A_0 \oplus \ell^2$. Similarly $\mathcal{T} \approx A_0 \oplus \ell^2$ and so the theorem now follows from Theorem 2.1. ∎

Remark 2.4: With respect to injections which map matrix units to sums of matrix units the Banach space $\mathcal{S}_{2^\infty}$ is not as injective as $\mathcal{T}_{2^\infty}$. More specifically if $\phi: M_2 \to T_{m_1}$ and $\phi': M_n \to T_{m_2}$ are isometric linear maps which map matrix units to sums of matrix units, and if $i: M_2 \to M_n$ is a unital C*-algebra injection, and if $\phi' \circ i = \sigma \circ \phi$, then it can be shown that $2n \leq m_1$. In particular there is an obstacle to constructing an injection $F \to \mathcal{S}_{2^\infty}$ along the lines of the proof of Lemma 1.1. This suggests that there are quite possibly *no* complemented injections of the Fermion algebra in $\mathcal{S}_{2^\infty}$, whereas it is clear that there are such injections for $\mathcal{T}_{2^\infty}$.

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